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SPIN AND ENERGY EFFECTS IN HEAVY QUARKONIUM SPECTROSCOPY

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Abstract: **Exact analytical solution of Schrodinger equation is solved using the spherically symmetric interquark potential consisting of harmonic oscillator term with linear energy dependence as the confinement part and the inverse square potential as the asymptotic part in nonrelativistic frame of reference. Spin dependent heavy quark spectroscopy is obtained by adding Breit-Fermi correction term to the interaction for $c\bar{c}$ and $b\bar{b}$ system. Further predicted values of fine and hyperfine splitting of S and P states are compared with the results of recent experiment and other authors. According to our calculation, hyperfine splitting of 1S and 2S and their ratio are of $b\bar{b}$ system is close to experimental value as showing importance of asymptotic term added in energy dependent potential.**

Keywords: *Quarkonium static potential, Breit- Fermi interaction, fine hyperfine splitting PACS No: 12.40Yx, 13.20 Gd*

I INTRODUCTION

The investigation of properties of mesons composed of heavy quark and antiquark throw important light into heavy quark dynamics. Heavy flavour $c\bar{c}$ and $b\bar{b}$ meson have rich spectroscopy with narrow states of charmonium and bottonium below threshold. These systems are analogous to positronium system. In the recent years, a number of new states have been claimed in experiment, several of which cannot be reconciled with the prediction of simple quark potential models. Theoretically potential models are used for deeper understanding of strong interaction between quark antiquark. Many potential models have been designed to reproduce spin dependent meson spectroscopy but have their limits. In earlier studies it is found that mass spectra shows saturation effect which means that as the quantum numbers increase the energy eigen values increase an upper bound. Our potential is motivated by this fact. Original form of energy dependent potential was used by Lombard [1] consisting of only confinement part. In our previous study [2], this potential is extended for asymptotic part also and formulated to describe the properties of quarkoniua. The experimental spectra of $c\bar{c}$ and $b\bar{b}$ systems exhibit fine structure and hyperfine splitting. The difference between energy of triplet ($^3S_1$) and singlet ($^1S_0$) for $c\bar{c}$ system is around 100 MeV and 50 MeV for n=1 and 2 respectively. These are estimated by adding Breit Fermi relativistic correction to the interaction. The Schrodinger equation with the complete potential has been solved exactly for radially and orbitally excited state of the system and the eigen values and eigen functions are calculated. The spin hyperfine interaction of vector and pseudoscalar states, are important property of quarkonia that provide better understanding of quark antiquark dynamics within the meson,

The paper is organized as follows. The Mathematical formulation and details of calculation have been given in section II. Results are summarized in section III. Finally in section IV, summary and conclusions are given.

II MATHEMATICAL FORMULATION AND DETAILS OF CALCULATION

A. Mass Spectra

A conventional meson can be described by the wave function of the bound quark antiquark state, which satisfies non-relativistic Schrodinger equation with chosen spherically symmetric potential

$$V(r, E_{n,l}) = \frac{1}{2} m\omega^2 r^2 (1+\gamma E_{n,l}) + \frac{g}{r^2} \qquad (1)$$

Where ω, γ and g are constants.
The three dimensional Schrodinger equation in the center - of - mass system is

$$\left[-\frac{\hbar^2}{2\mu}\overrightarrow{\nabla^2}+V(r,E_{n,\ell})\right]\psi_{n,\ell,m}(\vec{r})=E_{n,l}\psi_{n,\ell,m}(\vec{r})$$

, (2)

Where the reduced mass μ in terms of quark mass m_q and antiquark mass $m_{\bar{q}}$ is

$$\mu=\frac{m_q m_{\bar{q}}}{m_q+m_{\bar{q}}}$$

In natural units $\hbar=c=1$ is considered.

The wave function is written as

$$\psi_{n,\ell,m}(r,\theta,\phi)=\frac{u_{n,l}(r)}{r}Y_{\ell,m}(\theta,\phi)$$

Now putting $\omega'^2=\omega^2(1+\gamma E)$, $t=m\omega' r^2$, $\alpha=\frac{2E}{\omega'}$ and , $\beta=\ell(\ell+1)+2\mu g$, equation (2) reduces to

$$tu''(t)+\frac{u'(t)}{2}+\frac{1}{4}\left[\alpha-\frac{\beta}{t}-t\right]u(t)=0$$

(3)

u (t)s are the solution of the radial equation, which are bounded at infinity and are zero at the origin. As t tends to ∞ , the bounded solution behaves like exp(-t/2) and since t=0 is a singularity of equation (3) we seek for a solution in the form,

$$u(t)=t^{\eta}\exp\left(\frac{-t}{2}\right)R_{\ell}(t)$$,

(4)

in which, on account of boundary condition, η has to be positive. Substituting equation (4) into equation (3) and taking

$$\eta=\frac{1}{4}\left[1+\sqrt{(2\ell+1)^2+8\mu g}\right],$$

(5)

Resulting equation leads to the following equation

$$tR''_{\ell}(t)+\left[2\eta+\frac{1}{2}-t\right]R'_{\ell}(t)-\left[\eta+\frac{1}{4}-\frac{\alpha}{4}\right]R_{\ell}(t)=0$$

(6)

Apart from the constant factor the nonsingular solution of equation (6) is the Confluent Hypergeometric series [24]

$$R_{\ell}(t)=F\left[\eta+\frac{1}{4}-\frac{\alpha}{4},2\eta+\frac{1}{2};t\right]$$.

(7)

u(t) increases without bound as t→∞ unless the series F reduces to a polynomial. This occurs only if

$$\eta+\frac{1}{4}-\frac{\alpha}{4}=-n_r;n_r=0,1,2...,$$

(8)

which implies that the energy eigenvalue

$$E_{n,\ell}=-\frac{1}{8}a^2\omega^2\gamma+\frac{a\omega}{8}\sqrt{a^2\omega^2\gamma^2+16}$$

(9)

where $a=4n-2+\sqrt{(2\ell+1)^2+8\mu g}$

$E_{n,\ell}$ are the eigenenergies classified by principal and angular quantum numbers n and ℓ where $\ell\leq$ (n-1). The quark mass is connected to the physical mass as $M_{(q\,\bar{q})}=2m_q+E_{1s.}$

The method of calculating the leading relativistic correction to the energy spectrum of quarkonia is to add Breit- Fermi interaction (H_{BF}). It consists of three types of spin dependent interaction terms viz. spin spin, the spin orbit and tensor part given as [25]

$$V_{BF}(r)=V_{\ell S}(r)(\ell.\vec{S})+V_T(r)\left[S(S+1)-\frac{3(\vec{S}.\vec{r})(\vec{S}.\vec{r})}{r^2}\right]+V_{SS}(r)\left[S(S+1)-\frac{3}{2}\right]$$

(10)

Where ℓ is relative orbital angular momentum, S the total spin of two quarks.
H_{BF} can be generated from the scalar and vector gluon exchange components of the interquark potential
For the potential used in the present work
$V(r, E_{n,l})=V_V(r)+V_S(r)$,
where

$$V_V(r)=-\frac{g}{r^2} \quad \text{and} \quad V_S(r)=\frac{1}{2}m\omega^2(1+\gamma E_{n,\ell})r^2$$

(11)

Thus Fine structure of P mesonic state is described by radially dependent potential functions the spin orbit term $V_{\ell S}(r)$ and tensor term $V_T(r)$ while the spin singlet triplet hyperfine splitting are described by the term $V_{SS}(r)$ given as

$$V(\ell S) = \frac{1}{2m^2{}_q r}\left[3\frac{dV_V}{dr} - \frac{dV_S}{dr}\right] = \frac{1}{m_q{}^2}\left[\frac{3g}{r^4} - k\right]$$

(12)

where $k = \frac{1}{2} m\omega^2 (1 + \gamma E_{n,\ell})$,

$$V_T(r) = \frac{1}{12m^2{}_q}\left[\frac{1}{r}\frac{dV_V}{dr} - \frac{d^2V_V}{dr^2}\right] = \frac{2g}{3m^2{}_q r^4}$$

(13)

$$V_{SS}(r) = \frac{1}{6m^2{}_q}\left[\frac{d^2V_V}{dr^2} + \frac{2}{r}\frac{dV_V}{dr}\right] = -\frac{g}{3m^2{}_q r^4}$$

(14)

Assuming a, b and c correspond to expectation values of potential functions $V_{lS}(r)$, $V_T(r)$ and $V_{SS}(r)$ respectively, the fine hyperfine level splitting can be calculated by the following mass formulae for different orbital states.

(i) For S states

$$M(n^3S_1) = M_{n,S} + \frac{1}{4}c$$

(15)

$$M(n^1S_0) = M_{n,S} - \frac{3}{4}c$$

(16)

(i) For P states

$$M(n^3P_2) = M_{n,P} + a - \frac{2b}{5}$$

(17)

$$M(n^3P_1) = M_{n,P} - a + 2b$$

(18)

$$M(n^3P_0) = M_{n,P} - 2a + 4b$$

(19)

$$M(n^1P_1) = M_{n,P} - \frac{3}{4}c$$

(20)

Where $M_{n,P}$ ($M_{n,S}$) are spin average masses for the ℓ orbital (S spin) state of quarkonium obtained from the exact numerical solution.

The hyperfine splitting is defined as

$$\Delta M^S_{hyp} = M(n^3S_1) - M(n^1S_0) \quad \text{for S state}$$

(21)

$$\Delta M^P_{hyp} = M_{c.o.g.} - M(n^1P_1) \quad \text{for P state}$$

(22)

where $M_{c.o.g.}$, the center of gravity of the triplet states is

$$M_{c.o.g.} = (5M_{^3P_2} + 3M_{^3P_1} + M_{^3P_0})/9$$

(23)

According to Rayleigh-Schrodinger perturbation theory in the first order approximation [26, 27] the hyperfine splitting is also related to the square of modulus of wave function at the origin and dependent on the strong coupling constant α_S as

$$\Delta E_{HFS} = \frac{32\pi\alpha_S}{9m_q m_{\bar{q}}}|\psi(0)|^2$$

(24)

Also P wave splitting is characterized by the parameter P wave splitting [28] is given by

$$\chi = \frac{M(^3P_2) - M(^3P_1)}{M(^3P_1) - M(^3P_0)}$$

(25)

b. **Leptonic decay width of heavy qaurkonia:** The leptonic decay width of system $M_{q\bar{q}} \rightarrow e^+e^-$ is calculated according to Van Royen- Weisskoph formula [29].

$$\bar{\Gamma}(n^3S_1 \rightarrow e^+e^-) = \frac{4\alpha^2 e_q{}^2}{M_{ns}{}^2{}_{(q\bar{q})}}|R_{n,s}(0)|^2$$

(26)

Where $|R_{n0}(0)|$ is the nonvanishing radial wave function of S wave at the origin. M_{ns} is the mass of bound triplet (vector) state, α is the electromagnetic fine structure constant and e_q the charge of quark in units of the electron charge (e_c=2/3e, e_b= -1/3e). This formula is true for energy dependent potential also.

Details of Calculation: Firstly we obtain spin averaged mass spectra without introducing the Breit Fermi correction. The parameters ω, γ and g are obtained by fitting theoretically estimated values of E_{2S}, E_{3S} and center of gravity of E_{1P} with respect to E_{1S} to the experimental data [10]. These in turn are used to predict eigenvalues of higher excited levels by solving the energy eigenvalue equation (9). In the literature, the charm quark mass is chosen between 1.2 $<m_c<$1.8 GeV whereas that of the bottom quark is between 4.5< m_b < 5.4 GeV. In the present work we have chosen the mass to be 1.5 GeV for the charm quark and 5.0 GeV for the bottom quark, which is almost at the mid values of above ranges. While solving non linear energy eigenvalue equation (9) only negative roots of γ are accepted because negative values of γ can compress the spectrum, a feature which is observed experimentally. In the present work we have taken strong coupling constants $\alpha_s (m^2_c) = 0.37$, $\alpha_s (m^2_b) = 0.26$ according to Particle Data Group 2010 [10] by well measured hyperfine splitting of 1S.

Once the parameters of the potential are fixed it is solved numerically with reduced radial Schrodinger equation in MATHEMATICA 8.0 by software program obtained by LUCHA et al [11] for each quantum state separately. Using the spin dependent potential given by equation (10) eigenvalues of dimensionless Schrödinger equation is solved by numerical integration from -2 to 2 GeV in the steps of 0.1 GeV for each quantum state separately.

III RESULT AND DISCUSSIONS

Potential parameters obtained from the fit to the experimental data are shown in table 1. Full spin dependent spectra by adding Briet -Fermi correction term of radially and orbitally excited states are shown in **fig 1 and 2** for $c\bar{c}$ and $b\bar{b}$ families respectively and compared with those of experimental values [10] wherever available. Thresholds for strong decay are at $2m_D$=3.729 GeV for $c\bar{c}$ and at $2m_B$=10.574 GeV for $b\bar{b}$ quarkonia respectively shown as continuous dotted lines in the figures.

Hyperfine splitting for S and P states are given in **table 2 and table 3** respectively**.** Also given for the comparison are the experimental data [10], values calculated from equation (24) and theoretical prediction [12] for S states and those of Hayask et al [13] for P waves. Striking feature for $b\bar{b}$ system is that values of ΔM_{2S} predicted by our potential is almost consistent with the average of experimental results 24.3 MeV and 48.7 MeV obtained by Belle collaboration [14] and Dobbs et al. [15] respectively. Moreover ΔH_{1S} =61MeV calculated by equation (24) is consistent with the result predicted by lattice QCD [16, 17].

The ratio of 2S and 1S hyperfine splitting of $b\bar{b}$ system calculated by our potential model (35/65= 0.53) is almost equal to the average of two experimental result obtained by Belle[14] and Dobbs et al.[15] as 0.41and 0.72 respectively.

The contribution of the mass arising from fine splitting for 1P and 2P states for $c\bar{c}$ and $b\bar{b}$ systems are presented in **table 4**. The calculated values of ΔM_P (difference between center of gravity of ^3P_J and $^1P_1$ state) for $c\bar{c}$ is - 2.0 MeV against the experimental value of -0.9 MeV [18]. Our estimate is closer to the experimental data compared to those obtained by other authors [19, 20, 21]. The predicted mass of $^1P_1$ state of $b\bar{b}$ system is 9.853 GeV which lies between ^3P_J and $^3P_2$.

The ratio of mass splitting χ_c **[22]** calculated from the equation (25) is shown in **table 5** for both the systems and compared with the experimental value [10]. The splitting rises slowly in going from 1P to higher radial excitation, a trend also reflected in the experimental data of $b\bar{b}$ system. It is pertinent to mention that interquark potentials which do not have asymptotic term exhibit the opposite trend [23].

As the hyperfine splitting is sensitive to the short range behavior of the potential, this feature can be attributed to the asymptotic component of the potential.

IV SUMMARY AND CONCLUSION

A special class of energy dependent potentials has been used to obtain the mass spectrum of $c\bar{c}$ and $b\bar{b}$ systems in the framework of non-relativistic quantum mechanics. Energy dependent potential employed in the earlier study of Lombard et al comprised of only the confinement term of harmonic oscillator potential with small linear energy dependence. Main drawback of the Lombard potential is that it does not have any asymptotic form. It is well known that any general form of quark interquark potential should have an asymptotic term and a confinement term. An asymptotic term is important to account for the low energy spectrum and the short range behavior of the wave function. In view of this, in the present work we have added an inverse square potential as the asymptotic term. With this particular form of potential Schrödinger equation is exactly soluble. After fixing the potential parameters by fitting the lower energy levels with the experimental data Fermi Breit, correction is added to the potential to obtain the complete spin dependent spectra. The potential is different from each quantum state. The Schrödinger equation is solved numerically for each state to obtain the eigen values and eigen functions. The mass spectra so obtained are in good agreement with the experimental data wherever available. The energy dependent factor provides small perturbation to harmonic oscillator and the spacing of levels deviate from equal spacing. γ gives long distance contribution to level spacing and makes the confinement force weaker than harmonic oscillator. The main feature of introducing energy dependence is saturation effect in eigenvalues. The degree of saturation is determined by energy dependent factor. This saturation can be attributed to wave function at origin (WFO) increases as one goes from $c\bar{c}$ to $b\bar{b}$ (increasing reduced mass) [23] . Thus the heavier reduced mass system tends to spend lesser time in the deeper attractive region.

Our results for hyperfine splitting are closer to the experimental data compared to the values obtained by other authors. The reason being both these quantities depend directly on the square of the modules of the wave function at the origin, which have been calculated exactly in our work. This shows the importance of the asymptotic term in the antiquark potential in determining the wave function at the origin. As these results are sensitive to the quarkonium wave function, their successful prediction along with the spectroscopic prediction becomes an important measure of success of any theoretical model employed for the study of the quarkonia.

Table 1: Spectroscopic parameters obtained from fit to experimental data for 2S 3S and 1P with respect to 1S states.

	ω fm^{-1}	γ GeV^{-1}	G GeV^{-1}
$c\bar{c}$	0.233	-0.117	-0.194
$b\bar{b}$	0.187	-0.102	-0.044

Table 2: Hyperfine splitting of S wave in MeV of $c\bar{c}$ and $b\bar{b}$ systems

States	$c\bar{c}$				$b\bar{b}$			
	Our	**eq (24)**	**expt [10]**	**[14]**	**Our**	**eq (24)**	**expt [10]**	**[14]**
1^3S_1-1^1S_0	115	110	114	116	65	61	63	70
2^3S_1-2^1S_0	40	38	47	89	35	29	24	35
3^3S_1-3^1S_0	33	30	-	8	31	25	-	29
4^3S_1-4^1S_0	28	25	-		29	21	-	

Table 3: Hyperfine splitting P wave in MeV of $c\bar{c}$ and $b\bar{b}$ systems

	$c\bar{c}$			$b\bar{b}$		
States	Calc	[13]	expt [10]	Calc	[13]	expt [10]
1^3P_1-1^1P_1	-19	-10	-16	-5	-3.4	-12.21
2^3P_1-2^1P_1	-13	-	-	-4	-3.5	-

Table 4: Fine splitting of $c\bar{c}$ and $b\bar{b}$ systems in (MeV)

	ΔM	Our result	[19]	[20]	[21]	[10]
$c\bar{c}$ system						
1P	$M(^3P_2)-M(^3P_1)$	40	51	45	79	45.54
	$M(^3P_1)-M(^3P_0)$	69	83	91	98	95.91
	$M(^3P_2)-M(^3P_0)$	109	134	137	178	141.45
$b\bar{b}$ system						
1P	$M(^3P_2)-M(^3P_1)$	13	31	24	20	19.43
	$M(^3P_1)-M(^3P_0)$	20	41	37	25	33.34
	$M(^3P_2)-M(^3P_0)$	33	72	61	46	52.77
2P	$M(^3P_2)-M(^3P_1)$	12	24	17	12	13.19
	$M(^3P_1)-M(^3P_0)$	17	32	26	15	22.96
	$M(^3P_2)-M(^3P_0)$	29	56	44	07	36.15

Table 5: Ratio of mass splitting χ for ℓ=1, S=1 states of $c\bar{c}$ and $b\bar{b}$ system

	$c\bar{c}$		$b\bar{b}$	
	Our work	Exp [10]	Our work	Exp [10]
1P	0.57	0.49	0.67	0.57
2P	0.63	-	0.70	0.67
3P	0.67	-	0.75	-

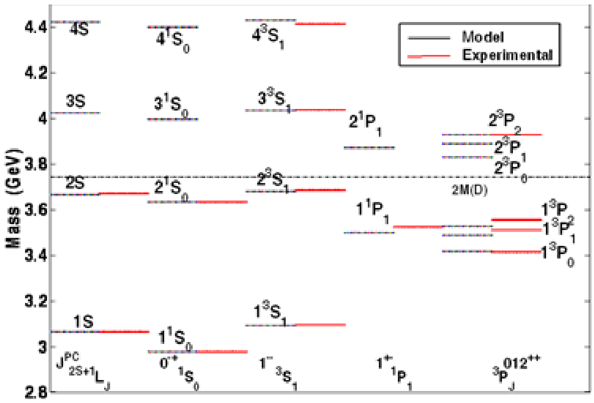

Figure 1: Spin dependent mass spectrum of $c\bar{c}$ quarkonia. Solid lines denote the levels which have been detected experimentally and dashed lines indicate the levels predicted by the potential model.

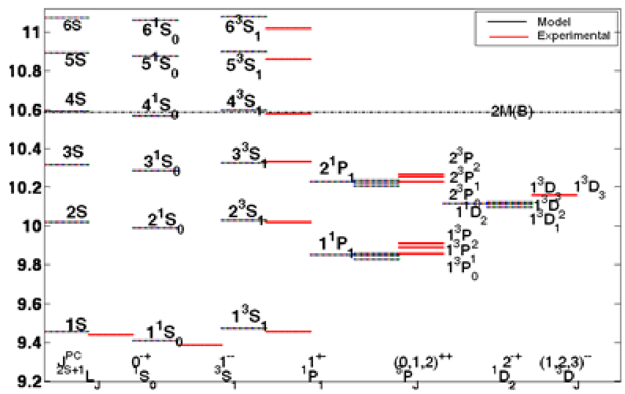

Figure 2: Spin dependent mass spectrum of $b\bar{b}$ quarkonia. Solid lines denote the levels which have been detected experimentally and dashed lines indicate the levels predicted by the potential model.

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